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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

The Journal is open to contributions from members as well as subscribers. The Editors may also accept contributions from others.

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If we replace this by the equation $w = 2h + 5c - 170$ which is much easier to handle, we shall not be making a very serious error, the errors being of the order ·6 and ·9 for the two pairs of extreme values of h and c.

6. Some writers, notably writers on Medical Jurisprudence, have suggested the use of the formula $w = \frac{hc}{17}$. We tried to see if we could derive this formula by fitting a regression plane to the logarithms of height, weight and chest-girth respectively, for $\log w = \log c + \log h - \log 17$ and if the fit by the equation $w = \frac{hc}{17}$ is very good, it is reasonable to expect the above equation to be the best fitting regression equation for the logarithm of weight in terms of the logarithms of height and chest-girth. So, from the original data we recorded the logarithms of these three factors, and constructed three correlation tables of these logarithms taken two by two, and calculated the regression of the logarithm of weight. The computed regression equation was

$$\log w = .9966 \log h + 1.3741 \log c - 1.7999845$$

$$\therefore \qquad w = \frac{hc^{1.37}}{63.093} \text{ approximately.}$$

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We next compared the three equations

$$(1) \qquad w = 2h + 5c - 170$$

$$(2) \qquad w = \frac{ch}{17}$$

$$(3) \qquad w = \frac{hc^{1.37}}{63.093}$$

by examining the fit in each of these three cases in a random sample of 150 picked out from the original records.

The goodness of fit of each of these formulae with the observed values of weight of the hundred and fifty individuals was also examined

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and the following result was obtained :—

Formula.	$2h + 5c - 170$	$\frac{ch}{17}$	$\frac{hc^{1.37}}{63}$
Goodness of fit P	.6323	.0080	.3342

Thus the best fit is given by the formula

$$w = 2h + 5c - 170.$$

$w = \frac{ch}{17}$ gives a very bad fit. On examining the fit a little more carefully it was seen to be very bad at the upper end, though it was not so bad in the middle, but a formula which gives good values for the weights of 'average individuals' but which completely fails in the case of 'extreme individuals' is surely not going to be of much use in practice. The formula $\frac{hc^{1.37}}{63}$ does not give a bad fit, but it is not easy to handle. We recommend $w = 2h + 5c - 170$ for general use since it is not only easy to handle but gives a good fit for both the average and the extreme type of individuals. The estimates of weight by this formula have a probable error of nearly 5.4 lbs.

Summary.

From a random sample of 1000 males for each of whom the height in inches, weight in pounds and chest-girth in inches were recorded three separate distributions of these characteristics were made, and curves were fitted to these distributions. A Pearson Type VII was fitted to the distribution of height and Pearson Type IV was fitted to both weight and chest-girth. The following results were obtained :—

(1) Average ht. = 5 ft. 4.95 in. S. D. of ht. = 2.31 in.

(2) Average wt. = 125.46 lbs S. D. of wt. = 16.43 lbs.

(3) Average chest-girth = 33.09 in. S. D. of chest-girth = 2.41 in.

Next these characteristics were taken two by two and their correlations calculated.

$$r_{hw} = .5144 \pm .0157,$$

$$r_{hc} = .3063 \pm .0193,$$

$$r_{wc} = .8301 \pm .0066.$$

Our next line of inquiry was to find the regression of weight on height and chest-girth. A regression formula $w = 2h + 5c - 170 \pm 5.4$ for weight in terms of height and chest-girth was obtained; and this was compared with the formula $w = \frac{ch}{17}$ advocated by some people, and also with another formula which we obtained by finding the regression of the logarithm of weight on the logarithms of height and chest-girth respectively. By far the best fit was given by $w = 2h + 5c - 170 \pm 5.4$, and this is moreover the easiest to handle, and we regarded this in many respects as superior to the other two, and we commended it for general use.

In conclusion, we have to thank the Manager of the United India Life Assurance Company for giving us free access to their records.

1. Correlation table connecting

Height

Weight	4 ft. 8.4 —9.4 in.	10.4	11.4	5 ft, 0.4 in.	1.4	2.4	3.4	4.4	5.4	6.4
80.4 —87.4		1		1					1	
—94.4				2	1	3	2		1	1
—101.4	1	1	1	4	4	8	8	8	1	4
—108.4			2	5	3	16	13	9	10	5
—115.4			2	2	7	18	41	26	24	18
—122.4				3	4	14	24	38	43	30
—129.4					1	10	16	34	27	43
—136.4			1	1	4	7	11	21	27	35
—143.4				1	1		6	5	14	20
—150.4							1	7	10	9
—157.4							1		5	5
—164.4								1	2	5
—171.4									1	1
—178.4									1	1
—185.4										3
—192.4										
—199.4										
—206.4										
—213.4										
—220.4										
Total.	1	2	6	19	25	76	123	149	167	180

Height

7.4	8.4	9.4	10.4	11.4	6 ft. 0.4 in	1.4	2.4	3.4	4.4	Total.
										3
										10
1	1	1								43
	1									64
10	3									151
16	15	2								189
23	14	6	2	1		1				178
28	9	9	3							156
12	12	2	3		2	1				81
9	14	4	1							55
7	1	3	1							23
7	4	3	1		1		1			25
3	1	1	1							8
		1	1			1			1	6
	2	1								6
1										1
1										1
118	77	33	13	3	3	3	1		1	1000

$r = 0.5144 \pm 0.0157.$

Height

Chest

	4 ft. 8.4—9.4 in	10.4 in.	11.4 in.	5 ft. 0.4 in.	1.4 in.	2.4 in.	3.4 in	4.4 in.	5.4 in	6.4 in.
25.4 in. 26.4 in.					1					
—27.4									1	
—28.4					1	1			1	1
—29.4		2		2	1	6	2	4	3	5
—30.4				6	1	14	17	15	8	8
—31.4	1		2	2	3	13	17	22	21	19
—32.4				4	6	16	32	35	40	28
—33.4			2	4	3	12	29	28	31	46
—34.4					2	6	8	13	18	27
—35.4			2	1	5	5	8	14	21	16
—36.4						1	7	8	12	12
—37.4					2	1	2	4	3	7
—38.4						1		3	3	5
—39.4							1	1	3	
—40.4								2	1	3
—41.4									1	
—42.4										3
—43.4										
—44.4										
—45.4										
Total	1	2	6	19	25	76	123	149	167	180

Height

7.4 in.	8.4 in.	9.4 in.	10.4 in.	11.4 in.	6 ft. 0.4 in.	1.4 in.	2.4 in.	3.4 in.	4.4 in.	Total.
										1
										1
1										5
										25
4	3	1								77
13	6	2								121
14	13	6	1			1				196
22	13	4	4	2	2					202
20	10	6	2	1	1					114
15	10	4	1			1	1		1	105
8	10	3	2			1				64
9	5	3	2							38
7	2	2								23
2	1									8
1	3	1								11
			1							2
										3
2										2
		1								1
	1									1
118	77	33	13	3	3	3	1		1	1000

$r = 0.3063 \pm 0.0193$

Chest

Weight	25.4 in. 26.4 in.	27.4 in.	28.4 in.	29.4 in.	30.4 in.	31.4 in.	32.4 in.	33.4 in	34.4 in.	35.4 in.
80.4 – 87.4				3						
– 94.4		1	2	5		1		1		
–101.4			2	8	14	10	6	3		
–108.4	1			1	20	18	20	4		
–115.4				4	23	43	47	26	5	3
–122.4			1	3	12	28	59	60	18	7
–129.4					5	14	49	51	34	16
–136.4				1	3	6	13	40	34	37
–143.4						1	1	12	16	26
–150.4							1	4	5	12
–157.4									2	3
–164.4								1		
–171.4										
–178.4										1
–185.4										
–192.4										
–199.4										
–206.4										
–213.4										
–220.4										
Total.	1	1	5	25	77	121	196	202	114	105

Chest

36.4 in.	37.4 in.	38.4 in.	39.4 in.	40.4 in.	41.4 in.	42.4 in.	43.4 in.	44.4 in.	45.4 in.	Total.
										3
										10
										43
										64
										151
	1									189
6	2	1								178
14	5	3								156
16	6	3								81
16	10	5	1	1						55
8	4	2	1	2	1					23
2	8	7	3	4						25
1	2	2	3							8
1				2	1	1				6
				2		2		1	1	6
			1				1			1
							1			1
64	38	23	8	11	2	3	2	1	1	1000

$r = .8301 \pm 0.0066$

ON A CERTAIN POLYNOMIAL ANALOGOUS TO LOMMEL'S POLYNOMIAL.

By R. S. VARMA, M. Sc.,
Lecturer, Christ Church College, Cawnpore.

In a recent paper,* Dr. S. C. Mitra has shown that the Parabolic cylinder function, $D_n(z)$, satisfies the relation

$$D_{m+n}(x) + (-1)^m (n+1) R_{n, m-2}(x) D_n(x) + (-1)^m R_{n-1, m-1}(x) D_{n+1}(x) = 0, \qquad \ldots \quad (1)$$

where $R_{n,m}(x)$ is a polynomial analogous to that of Lommel† in the theory of Bessel Functions. Dr. Mitra has investigated certain interesting properties of $R_{n,m}(x)$. The object of the present paper is to obtain the relation (1) by a different method, which has the advantage of giving us the differential equation satisfied by the function $R_{n,m}(x)$. By the help of this differential equation $R_{n,m}(x)$ is at once expressed in a series of Hermite's polynomial $H_n(x)$.

The differential equation satisfied by $D_{m+n}(x)$ is

$$\frac{d^2 y}{dx^2} + (m + n + \tfrac{1}{2} - \tfrac{1}{4}x^2)\, y = 0. \qquad \ldots \quad (2)$$

Now, if (2) has a solution of the form

$$A(x) D_n(x) + B(x) D_{n+1}(x), \qquad \ldots \quad (3)$$

it is easy to see, by the help of the recurrence formulae,‡

$$D_{n+1}(x) - x D_n(x) + n D_{n-1}(x) = 0$$

and

$$D'_n(x) + \tfrac{1}{2} x D_n(x) - n D_{n-1}(x) = 0,$$

---

* "On the properties of a certain polynomial analogous to Lommel's polynomial," *Indian Physico-Mathematical Journal*, Vol. 3, No. 1 (1932), pp. 9—15.

† Watson : *Bessel Function*, § 9.6, pp. 294 *et seq.*

‡ Whittaker and Watson : *Modern Analysis* (5th Edition), p. 350.

that

$$A''(x) + x\,A'(x) + m\,A(x) + 2\overline{n+1}\,B'(x) = 0 \qquad \ldots (4)$$

and

$$B''(x) - x\,B'(x) + \overline{m-1}\,B(x) - 2\,A'(x) = 0 \qquad \ldots (5)$$

where accents denote differentiations with respect to x.

We deduce, therefore, that $A(x)$ and $B(x)$ must satisfy the differential equations

$$A^{iv}(x) + (2m+4n+4-x^2)\,A''(x) - 3x\,A'(x) + m\,(m-2)\,A(x) = 0 \quad \ldots (6)$$

and

$$B^{iv}(x) + (2m+4n+2-x^2)\,B''(x) - 3x\,B'(x) + (m^2-1)\,B(x) = 0 \ldots (7)$$

respectively.

We see that the equation (7) may be deduced from the equation (6) by putting $n-1$ for n and $m+1$ for m. Hence we shall solve only one of the equations (6) and (7) to find the nature of the functions $A(x)$ and $B(x)$ and this we shall do with the help of Hermite's polynomial $H_n(x)$ which is related to $D_n(x)$ by the relation

$$H_n(x) = e^{\frac{1}{4}x^2}\,D_n(x).$$

Since we must have

$$D_{m+n}(x) = A(x)\,D_n(x) + B(x)\,D_{n+1}(x), \qquad \ldots (8)$$

and since for positive integral values of n, $D_n(x)$ is a polynomial multiplied by $e^{-\frac{1}{4}x^2}$, $A(x)$ and $B(x)$ must be simple polynomials in x. For this reason we seek solution of (7) in the form $\Sigma\,B_r\,H_r(x)$. Substituting $\Sigma\,B_r\,H_r(x)$ in (7) and remembering that*

$$\frac{d}{dx}H_r(x) = r\,H_{r-1}(x),$$

and

$$H_r(x) - x\,H_{r-1}(x) + (r-1)\,H_{r-2}(x) = 0,$$

we come to the following relation between the co-efficients of consecutive terms

$$B_{r+2} = \frac{(r+1+m)(r+1-m)}{(r+1)(r+2)(2m+4n-2r-2)}\,B_r. \qquad \ldots (9)$$

---

* P. Appell and J. Kampe de Feriet: *Fonctions Hypergeometriques et Hyperspheriques, Polynomes d'Hermite*, pp. 335-6.

In a similar manner, for the equation (6) to have a solution of the form $\Sigma A_r H_r(x)$, we must have

$$A_{r+2} = \frac{(r+m)(r+2-m)}{(r+1)(r+2)(2m+4n-2r)} A_r \qquad \dots \quad (10)$$

From (9) and (10) it is clear that

$$A(x) = A_0 \left[H_0(x) + \frac{m(2-m)}{2!\,(2m+4n)} H_2(x) + \frac{m(2+m)(2-m)(4-m)}{4!\,(2m+4n)(2m+4n-4)} H_4(x) + \dots \right]$$

$$+ A_1 \left[H_1(x) + \frac{(1+m)(3-m)}{3!\,(2m+4n-2)} H_3(x) + \frac{(1+m)(3+m)(3-m)(5-m)}{5!\,(2m+4n-2)(2m+4n-6)} H_5(x) + \dots \right],$$

and

$$B(x) = B_0 \left[H_0(x) + \frac{(1+m)(1-m)}{2!\,(2m+4n-2)} H_2(x) + \frac{(1+m)(3+m)(1-m)(3-m)}{4!\,(2m+4n-2)(2m+4n-6)} H_4(x) + \dots \right]$$

$$+ B_1 \left[H_1(x) + \frac{(2+m)(2-m)}{3!\,(2m+4n-4)} H_3(x) + \frac{(2+m)(4+m)(2-m)(4-m)}{5!\,(2m+4n-4)(2m+4n-8)} H_5(x) + \dots \right]$$

As shown above $A(x)$ and $B(x)$ must be finite; hence of the four constants, A_0, A_1, B_0, and B_1, we choose, when m is even

$$A_1 = 0, \quad B_0 = 0,$$

and when m is odd

$$A_0 = 0, \quad B_1 = 0.$$

We have, therefore, when m is even,

$$D_{m+n}(x) = A_0 \Big[H_0(x) + \frac{m(2-m)}{2!(2m+4n)} H_2(x) + \frac{m(2+m)(2-m)(4-m)}{4!(2m+4n)(2m+4n-4)} H_4(x) + \dots \Big] D_n(x)$$

$$+ B_1 \Big[H_1(x) + \frac{(2+m)(2-m)}{3!(2m+4n-4)} H_3(x) + \frac{(2+m)(4+m)(2-m)(4-m)}{5!(2m+4n-4)(2m+4n-8)} H_5(x) + \dots \Big] D_{n+1}(x); \quad \dots (11)$$

and when m is odd,

$$D_{m+n}(x) = A_1 \Big[H_1(x) + \frac{(1+m)(3-m)}{3!(2m+4n-2)} H_3(x) + \frac{(1+m)(3+m)(3-m)(5-m)}{5!(2m+4n-2)(2m+4n-6)} H_5(x) + \dots \Big] D_n(x)$$

$$+ B_0 \Big[H_0(x) + \frac{(1+m)(1-m)}{2!(2m+4n-2)} H_2(x) + \frac{(1+m)(3+m)(1-m)(3-m)}{4!(2m+4n-2)(2m+4n-6)} H_4(x) + \dots \Big] D_{n+1}(x) \quad \dots (12)$$

Comparing the co-efficients of $e^{-\frac{1}{4}x^2} x^{m+n}$ and $e^{-\frac{1}{4}x^2} x^{m+n-2}$ in the above relations, we easily obtain that

$$A_0 = -\frac{(m-2)!(2m+4n)(2m+4n-4)\dots(2m+4n-2m+8)}{m(2+m)\dots(m-4+m)(2-m)(4-m)\dots(m-2-m)}(n+1);$$

$$A_1 = -\frac{(m-2)!(2m+4n-2)(2m+4n-6)\dots(2m+4n-2m+8)}{(1+m)(3+m)\dots(m-4+m)(3-m)(5-m)\dots(m-2-m)}(n+1);$$

$$B_0 = \frac{(m-1)!(2m+4n-2)(2m+4n-6)\dots(2m+4n-2m+4)}{(1+m)(3+m)\dots(m-2+m)(1-m)(3-m)\dots(m-2-m)};$$

and

$$B_1 = \frac{(m-1)!(2m+4n-4)(2m+4n-8)\dots(2m+4n-2m+4)}{(2+m)(4+m)\dots(m-2+m)(2-m)(4-m)\dots(m-2-m)}.$$

We therefore conclude that $R_{n,\,m-2}(x)$ is a polynomial which satisfies the differential equation (6) and on comparing (11) and (12) with (1), we obtain that, when m is even,

$$(-1)^m R_{n,\,m-2}(x)$$

$$= \frac{(m-2)!\,(2m+4n)\,(2m+4n-4)\,\ldots\,(2m+4n-2m+8)}{m\,(2+m)\,\ldots\,(m-4+m)\,(2-m)\,(4-m)\,\ldots\,(m-2-m)} \times$$

$$\left[H_0 + \frac{m\,(2-m)}{2!\,(2m+4n)}\,H_2(x) + \frac{m\,(2+m)\,(2-m)\,(4-m)}{4!\,(2m+4n)\,(2m+4n-4)}\,H_4(x) + \ldots\right]$$

and when m is odd,

$$(-1)^m R_{n,\,m-2}(x)$$

$$= \frac{(m-2)!\,(2m+4n-2)\,(2m+4n-6)\,\ldots\,(2m+4n-2m+8)}{(1+m)\,(3+m)\,\ldots\,(m-4+m)\,(3-m)\,(5-m)\,\ldots\,(m-2-m)} \times$$

$$\left[H_1 + \frac{(1+m)\,(1-m)}{2!\,(2m+4n-2)}\,H_3(x) + \frac{(1+m)\,(3+m)\,(1-m)\,(3-m)}{4!\,(2m+4n-2)\,(2m+4n-6)}\,H_5(x) + \ldots\right]$$

A THEOREM ON CHARACTERS (II).

By S. Chowla.

In a previous paper*, I showed that if $\chi(n)$ is a real primitive character (mod k), then

$$L(1) = \sum_{1}^{\infty} \frac{\chi(n)}{n} = \Omega_{R}(\log \log k) \quad \ldots \quad (1)$$

$$\sum_{m=1}^{t} \chi(m) = \Omega_{R}(\sqrt{k} \log \log k) \quad \ldots \quad (2)$$

The first of these results was proved by Littlewood† *on the assumption of an unproved result*. The second is a stronger form of Paley's result‡

$$\sum_{m=1}^{t} \chi(n) = \Omega(\sqrt{k} \log \log k) \quad \ldots \quad (3)$$

The method used in (1) is entirely different from the methods of Littlewood and Paley, and uses the theory of the representation of a number as a sum of three squares. In this paper I use Paley's method to show that, *on the extended Riemann hypothesis*,

$$\sum_{m=1}^{t} \chi(n) = \Omega_{R}(\sqrt{p} \log \log p); \quad (4)$$

on the same hypothesis I show that § with $t < p/2$

$$\sum_{1}^{t} \chi(n) = \Omega_{L}(\sqrt{\log p}) \quad \ldots \quad (5)$$

where p runs only through primes and $\chi(n)$ is a character (mod p).

---

* To appear in the *Mathematische Zeitschrift*.

† *Proc. London Math. Soc.* 27 (1928), 358—372.

‡ *Jour. London Math. Soc* 7 (1932), 28—32.

§ I am indebted to Mr. Paley for a remark which shows that the result is trivial unless $t < p/2$.

1. Proof of (4).

Using the notation of Paley*, we have, when $\chi(-1) = +1$

$$\sum_{m=1}^{n-1} \chi(m) + \tfrac{1}{2}\chi(n) = \frac{\sqrt{k}}{\pi}\,\varepsilon(\chi) \sum_{m=1}^{\infty} \frac{\bar{\chi}(m)}{m} \sin\frac{2mn\pi}{k} \quad \ldots (6)$$

where $\varepsilon(\chi)$ is a constant of modulus 1 depending on χ. Now let

$$n \leqq \frac{k\theta}{2\pi} < n+1,$$

and we observe that

$$\sum_{m=1}^{k} \frac{\bar{\chi}(m)}{m}\left\{\sin m\theta - \sin\frac{2mn\pi}{k}\right\} = O\left(\sum_{m=1}^{k} \frac{1}{m}\cdot\frac{m}{k}\right) \quad \ldots (7)$$

$$= O(1).$$

Further, we have †,

$$\sum_{m=k+1}^{\infty} \frac{\bar{\chi}(m)}{m} \sin\frac{2mn\pi}{k} = O(1).$$

Combining (8), (6) and (7), we have

$$\sum_{m=1}^{n} \chi(m) = \frac{\sqrt{k}}{\pi}\,\varepsilon(\chi) \sum_{m=1}^{k} \frac{\bar{\chi}(m)}{m} \sin m\theta + O(\sqrt{k}).$$

Now let N be an arbitrary number less than k. Then, ϕ being fixed,

$$\sum_{m=1}^{N} \frac{\bar{\chi}(m)}{m}\sin m\phi = \sum_{m=1}^{k} \frac{\bar{\chi}(m)}{m}\sin m\theta + \sum_{m=1}^{N} \frac{\bar{\chi}(m)}{m}(\sin m\phi - \sin m\theta)$$

$$- \sum_{N+1}^{k} \frac{\bar{\chi}(m)}{m}\sin m\theta$$

$$= S_1 + S_2 - S_3(\theta).$$

* *loc. cit.* † Paley, *loc. cit.*

By the first mean-value theorem

$$| S_2 | \leqq \sum_{m=1}^{N} \frac{1}{m} \cdot m \, | \theta - \phi | = N \, | \theta - \phi | \quad \ldots \quad (11)$$

We now choose

$$\theta = \phi + \frac{\beta \log \log N}{N} \qquad (0 \leqslant \beta \leqslant 1) \quad \ldots \quad (12)$$

and β will be determined later.

Then

$$| S_2 | \leqq \log \log N. \quad \ldots \quad (13)$$

Again, since $\chi(m)$ is real,

$$\int_0^{2\pi} \left[\sum_{N+1}^{k} \frac{\chi(m)}{m} \sin mt \right]^2 dt = \pi \sum_{N+1}^{k} \frac{\chi^2(m)}{m^2} = O\left[\frac{1}{N}\right]$$

whence

$$\int_{\phi}^{\phi + \frac{\log \log N}{N}} \left[\sum_{N+1}^{k} \frac{\chi(m)}{m} \sin mt \right]^2 dt = O\left[\frac{1}{N}\right] \quad \ldots \quad (14)$$

It follows from (14) that there is a β $(0 \leqslant \beta \leqslant 1)$ such that (for large N), with $\theta = \phi + \dfrac{\beta \log \log N}{N}$,

$$\left(\sum_{N+1}^{k} \frac{\chi(m)}{m} \sin m\theta \right)^2 = O\left(\frac{1}{\log \log \log N} \right) \quad \ldots \quad (15)$$

for if not, then

$$\int_{\phi}^{\phi + \frac{\log \log N}{N}} \left(\sum_{N+1}^{k} \frac{\chi(m)}{m} \sin mt \right)^2 dt = \Omega\left(\frac{\log \log N}{N \log \log \log N} \right),$$

which contradicts (14).

From (10), (13) and (15) it follows that

$$\sum_{m=1}^{N} \frac{\bar{\chi}(m)}{m} \sin m\phi = \sum_{m=1}^{k} \frac{\bar{\chi}(m)}{m} \sin m\theta + O(\log \log N) \quad \ldots \quad (16)$$

From (9) and (16) it follows that

$$\sum_{m=1}^{n} \chi(m) = \frac{\sqrt{k}}{\pi} \varepsilon(\chi) \sum_{m=1}^{N} \frac{\bar{\chi}(m)}{m} \sin m\phi + O(\sqrt{k} \log\log N) \quad \ldots (17)$$

Now let $\phi = \frac{\pi}{2}$. It follows that

$$\sum_{m=1}^{n} \chi(m) = \frac{\sqrt{k}}{\pi} \varepsilon(\chi) \sum_{m=1}^{N} \frac{\chi(m)}{m} \sin \frac{m\pi}{2} + O(\sqrt{k} \log\log N), \quad \ldots (18)$$

We now let $\chi(m) = \left(\frac{m}{p}\right)$ where p denotes a prime which is determined as follows. Let α_r denote any quadratic residue of p_r, the rth prime, and let β_r denote any non-residue of p_r. Let p be the least positive prime satisfying the congruences

$$p \equiv 1 \pmod 4$$

$$p \equiv 2 \pmod 3$$

$$p \equiv \alpha_r \pmod{p_r} \text{ if } p_r \equiv 1 \pmod 4$$

$$p \equiv \beta_r \pmod{p_r} \text{ if } p_r \equiv 3 \pmod 4$$

Then
$$\left(\frac{p_r}{p}\right) \equiv +1 \text{ if } p_r \equiv 1 \pmod 4$$

$$\left(\frac{p_r}{p}\right) \equiv -1 \text{ if } p_r \equiv 3 \pmod 4.$$

Obviously $p = 4p_2p_3 \ldots p_r u + S$, where S is the least positive integer satisfying the above relations and obviously $S < 4p_2p_3 \ldots p_r$, where u is variable and $(S, 2p_2 \ldots p_r) = 1$. Now Titchmarsh* has shown that for $k < x$, $(k, l) = 1$,

$$\sum_{\substack{p \leqq x \\ p \equiv l \pmod k}} 1 = \frac{1}{\phi(k)} \int_2^x \frac{du}{\log u} + O(x^{\frac{1}{2}} \log x)$$

---

* *Rendiconti del Circolo Matematico di Palermo*, **LIV** (1930), 414, 429. Theorem 6.

From this theorem it follows that if $(k, l) = 1$ then there is a prime $p = kx + l$ with $p < k^3$ (for large k). Hence as $r \to \infty$ there is a prime $p = 4\, p_2\, p_3 \ldots p_r\, u + S$ and $p < 64\, (p_2 \ldots p_r)^3$, whence

$$\log p < \log 64 + 3\,(\log p_2 + \ldots + \log p_r) \sim 3\, p_r \qquad \ldots \quad (19)$$

Now let $N = p_r$ in (17), $k = p$. Then $\varepsilon\,(\chi) = +1$. And taking

$$\chi\,(m) = \left(\frac{m}{p}\right),$$

we have

$$\sum_{m=1}^{n} \chi\,(m) \sim \frac{\sqrt{p}}{\pi} \sum_{m=1}^{p_r} \frac{\chi\,(m)}{m} \sin \frac{m\pi}{2} + O\,(\sqrt{p} \log\log p_r)$$

$$= \frac{\sqrt{p}}{\pi} \sum_{\substack{m \equiv 1 \pmod 2 \\ m \leqq p_r}} \frac{1}{m} + O\,(\sqrt{p} \log\log p_r)$$

$$\sim \frac{\sqrt{p}}{2\pi} \log p_r$$

$$> \frac{\sqrt{p}}{2\pi}\,(\tfrac{1}{2} \log\log p)\,.$$

Hence, for infinitely many large primes p it is true that

$$\sum_{m=1}^{n} \left(\frac{m}{p}\right) > \frac{\sqrt{p}}{5\pi} \log\log p$$

where

$$n = \frac{p}{4} + O\left(p\,\frac{\log\log\log p}{\log p}\right),$$

and this proves (4).

2. Proof of (5).

Let $\qquad \lambda\,(n) = (-1)^{\alpha_1 + \alpha_2 + \ldots + \alpha_r}$

where $\qquad n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}.$

It is easy to show that*

$$L(x) = \sum_1^x \lambda(n) = \Omega_L(\sqrt{x}) \qquad \ldots \quad (20)$$

Now let $\chi(n) = \left(\frac{n}{p}\right)$ where p is a prime determined by the congruences

$$p \equiv 5 \pmod 8$$

$$p \equiv \beta_r \pmod{p_r}$$

where β_r is any quadratic non-residue of p_r. Then it is easy to see that $\chi(n) = \lambda(n)$ for $n \leqq p_r$.

Now let p_r be the least prime $> x$, where x satisfies (20). Then, by the Prime Number Theorem, $p_r \sim x$. Hence

$$\sum_1^x \chi(n) = \Omega_L(\sqrt{x}) = \Omega_L(\sqrt{p_r}) \qquad \ldots \quad (21)$$

Again $\log p < 4\,p_r$ as in (19).

Hence $$\sqrt{p_r} > \sqrt{\frac{\log p}{4}},$$

and hence from (21)

$$\sum_1^x \chi(n) = \Omega_L(\sqrt{\log p}). \qquad \ldots \quad (22)$$

This proves (5).

In conclusion, I express my thanks to Dr. Heilbronn of Göttingen for carefully reading through my manuscript.

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* Landau: *Vorlesungen uber Zahlentheorie*, S. 162, shows that

$$M(x) = \sum_1^x \mu(n) = \Omega(\sqrt{x});$$

(20) can be proved by the same method.

A THEOREM ON NORMAL RECTILINEAR CONGRUENCES.

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The object of this paper is to obtain both analytically and geometrically the following theorem :—

There exist ∞^2 ruled surfaces of an ordinary congruence the osculating quadrics of which are equilateral, but there are only ∞^1 such ruled surfaces if the congruence is normal.

Let a congruence of straight lines be defined by

$$\xi = x + tX, \quad \eta = y + tY, \quad \zeta = z + tZ,$$

where x, y, z; X, Y, Z are functions of two variables u and v.

If the congruence is normal, we have

$$\Sigma X_1 x_2 - \Sigma X_2 x_1 = 0 \qquad \ldots \quad \ldots \quad (1)$$

Taking the surface of reference as a normal surface, when the point (u, v) describes a curve on the surface, the lines (X, Y, Z) generate a ruled surface whose equations are

$$\xi = x + tX, \quad \eta = y + tY, \quad \zeta = z + tZ,$$

where now x, y, z are functions of one parameter s, say.

The condition that the osculating quadric of this ruled surface be equilateral is

$$\Sigma X'' x' - \Sigma X' x'' = 0.^* \qquad \ldots \quad (2)$$

where accents denote differentiations with regard to s.

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* See my paper on Equilateral Osculating Quadrics of Ruled Surfaces, § IV.

Now $X' = X_1 u' + X_2 v'$

$$X'' = X_{11}u'^2 + X_{22}v'^2 + 2X_{12}u'v' + X_1 u'' + X_2 v''$$

$$x' = x_1 u' + x_2 v'$$

$$x'' = x_{11}u'^2 + x_{22}v'^2 + 2x_{12}u'v' + x_1 u'' + x_2 v''.$$

Therefore (2) becomes

$$\Sigma \{ X_{11}u'^2 + X_{22}v'^2 + 2X_{12}u'v' + X_1 u'' + X_2 v'' \} x_1 u' + x_2 v')$$
$$- \Sigma \{ x_{11}u'^2 + x_{22}v'^2 + 2x_{12}u'v' + x_1 u'' + x_2 v'' \} (X_1 u' + x_2 v') = 0,$$

or $$u'^3 (\Sigma x_1 X_{11} - \Sigma x_{11} X_1) + u'^2 v' \{ 2 (\Sigma x_1 X_{12} - \Sigma X_1 x_{12}) + \Sigma x_2 X_{11} - \Sigma X_2 x_{11} \}$$
$$+ u'v'^2 \{ 2 \Sigma x_2 X_{12} - \Sigma X_2 x_{12}) + \Sigma x_1 X_{22} - \Sigma x_{22} X_1 \}$$
$$+ v'^3 (\Sigma x_2 X_{22} - \Sigma X_2 x_{22}) + (u'v'' - v'u'') (\Sigma x_1 X_2 - \Sigma X_1 x_2) = 0 \quad \ldots \quad (3)$$

Since the congruence is normal, the terms involving the second differential co-efficients vanish because $\Sigma\, x_1 X_2 - \Sigma\, X_1 x_2 = 0$, from (1). Therefore this is a differential equation of the first order in dv/du and hence represents a single infinity of curves on the surface, which determine ∞' ruled surfaces which possess equilateral osculating quadrics.

If the congruence is not normal but an ordinary one, the condition that the ruled surface has equilateral osculating quadrics is

$$\Sigma\, X'x'' - \Sigma\, X''x' + 2\, \Sigma\, Xx' . (\Sigma\, XX'' - \Sigma\, X'^2) = 0^*$$

or $$\text{L. H. S. of (3)} - 4 \{ u' . \Sigma\, Xx_1 + v' . \Sigma\, Xx_2 \} \times$$
$$\{ u'^2\, \Sigma\, X_1^2 + v'^2\, \Sigma\, X_2^2 + 2\, \Sigma\, X_1 X_2 . u'v' \} = 0 \quad \ldots \quad (4)$$

since $\Sigma\, XX' = 0$ and $\therefore \Sigma\, XX'' - \Sigma\, X'^2 = -2\, \Sigma\, X'^2$

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* See my paper on Equilateral Osculating Quadrics of Ruled Surfaces, § IV.

which is a differential equation of the second order in dv/du and hence gives ∞^2 curves on the surface, and hence ∞^2 ruled surfaces the osculating quadrics of which are equilateral. This proves the theorem.

Geometrical Proof.

We know that the foci of a line l of the congruence are the same for all the ruled surfaces through l. Also by Chasle's Law, $\tan \phi = u/\beta$ where u is the distance of a point on a generator measured from the central point, ϕ the angle between the tangent plane at the point and the central plane, and β the parameter of distribution. It follows that $\phi = 0$ when $u = 0$. Hence for all ruled surfaces through l, $\phi = 0$ at $u = 0$, which means that any two ruled surfaces through l touch each other at the foci of l. Now when two ruled surfaces of a congruence C touch along a line l they have common asymptotic directions at the foci of l which can be proved as follows:—

Suppose they have common asymptotic directions at any point Q, say, then they have contact of the second order at Q and hence the curve of intersection has a triple point at Q. Part of this curve is the line l counted twice. Hence there is another branch of this curve through Q. Now if P is any point on the branch near Q, the generators of the two ruled surfaces through P are two of the lines of the congruence C that pass through P. As P moves up to Q along this branch, these two lines of the congruence come into coincidence at Q and therefore by definition Q is a focus.

Now suppose R is a ruled surface through a line l of a normal congruence and suppose that R has an equilateral osculating quadric S, which touches it along l. Then since the congruence is normal, the focal planes of l are at right angles and form an equilateral osculating quadric S_2, say, touchng R along l. Since S_1 and S_2 are both equilateral osculating quadrics it follows that $S_1 + KS_2 = 0$, for all values of K, is also an equilateral osculating quadric, passing through the points

of intersection of S_1 and S_2, *i.e.*, containing the line l and the two common asymptotic directions at the two foci of l. Hence, if a particular ruled surface R through l has an equilateral osculating quadric which touches it along l, then every ruled surface of the congruence which touches R along l also has an equilateral osculating quadric, the tangents to the asymptotic lines of the ruled surface at the foci of l being the generators of the osculating quadric.

Hence there is a single infinity of ruled surfaces through a line of normal congruence, which have equilateral osculating quadrics.

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NOTES AND QUESTIONS.

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Notes and Questions.

On a Formula in Elementary Trigonometry.

It is usual to express $\sin^n\theta$ and $\cos^n\theta$ as sums of sines and cosines of multiples of θ by the aid of De Moivre's Theorem. An elementary method is given in Hobson's *Trigonometry* (§ 50). The following method is a simple variation of the latter.

We have

$$2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)=\Sigma\cos(\alpha\pm\beta) \quad \dots (1)$$

where Σ extends to the cosines of all angles obtained by changing in all possible ways the sign of β in $\alpha\pm\beta$.

Again $\quad 2^2\cos\alpha\cos\beta\cos\gamma$

$$=2[\Sigma\cos(\alpha\pm\beta)]\cos\gamma=\Sigma\cos(\alpha\pm\beta\pm\gamma) \quad \dots (2)$$

where the latter Σ includes cosines of all angles obtained by changing the signs of β, γ in all possible ways.

Similarly we may establish by induction that

$$2^{n-1}\cos\alpha\cos\beta\cos\gamma\dots=\Sigma\cos(\alpha\pm\beta\pm\gamma\dots) \quad \dots (3)$$

where $\alpha, \beta, \gamma \dots$ are any n angles.

By regrouping the terms, (3) can be written

$$2^{n-1}\cos\alpha\cos\beta\cos\gamma\dots=\Sigma_0+\Sigma_1+\Sigma_2+\dots+\Sigma_{n-1} \quad \dots (3')$$

where Σ_r stands for the sum of the cosines of angles $\alpha\pm\beta\pm\gamma\pm\dots$ in which r signs are negative while the rest are positive.

If we write θ for $\alpha, \beta, \gamma \dots$ in (3), we get

$$2^{n-1}\cos^n\theta=\cos n\theta+{}_{n-1}C_1\cos(n-2)\theta+{}_{n-1}C_2\cos(n-4)\theta$$

$$+\dots+{}_{n-1}C_{n-2}\cos(-n+4)\theta+{}_{n-1}C_{n-1}\cos(-n+2)\theta \quad \dots (4)$$

We now observe that

$${}_{n-1}C_{n-1}\cos(-n+2)\,\theta = {}_{n-1}C_0\cos(n-2)\,\theta\,;$$

$${}_{n-1}C_{n-2}\cos(-n+4)\,\theta = {}_{n-1}C_1\cos(n-4)\,\theta;$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\therefore \quad 2^{n-1}\cos^n\theta = \cos n\,\theta + ({}_{n-1}C_1 + {}_{n-1}C_0)\cos(n-2)\,\theta$$
$$+ ({}_{n-1}C_2 + {}_{n-1}C_1)\cos(n-4)\,\theta + \dots$$
$$= \cos n\,\theta + {}_nC_1\cos(n-2)\,\theta + {}_nC_2\cos(n-4)\,\theta$$
$$+ \dots \quad \dots \quad \dots$$

Apart from the use made in this note, formula (3) is so very easy to remember and so very useful that it deserves a prominent place in any text-book of Elementary Trigonometry.

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B. B. BAGI.

Hain's Theorem.

The following is an elegant proof of the theorem that if AD, BE, CF are the perpendiculars of a triangle ABC, the lines joining the middle points of AD, BC ; BE, CA ; CF, AB are concurrent in the symmedian point of ABC.

It is known that the locus of the centres of a system of conics inscribed in a quadrilateral is a straight line passing through the middle points of the three diagonals. Let two of the adjacent sides of the quadrilateral coalesce, so that we have a triangle ABC and a point D on the side BC, say, where BC is the coalesced pair of sides, D being the point of their ultimate intersection and AB, AC are the two remaining sides of the quadrilateral. The system of conics are the in-conics of the triangle ABC touching BC at a fixed point D, and the line of centres is the line joining the middle points of AD, BC. It follows that the centre K of an in-conic whose points of contact with the sides are D, E, F lies on the three lines joining the middle points of AD, BC ; BE, CA ; CF, AB Hain's theorem is a special case of this, when AD, BE CF are the perpendiculars of the triangle ABC for, then AK, BK, CK bisect EF, FD, DE which are antiparallels to BC, CA, AB respectively and hence K is the symmedian point of ABC.

A. RANGANATHA RAO.

On the Pedal Lines of a System of Concyclic Points.

The following theorem is an extension of the well-known fact that the feet of the perpendiculars on the sides of a triangle from any point on the circum-circle are collinear. Given four points on a circle C, they can be taken in sets of three in four ways ; the feet of the perpendiculars from any arbitrary point O on C on the four pedal lines of O arising therefrom are collinear. Calling this new line as the pedal line of O with respect to the four points, we may extend the theorem to five points on the circle. In general

> With respect to n given points on a circle C there is a pedal line of O, which contains the feet of the perpendiculars from O on the n pedal lines of the n sets of $n-1$ points each.

We prove this theorem, which is due to Langley, first by reciprocating a special case of the following one:

1. Associated with four coplanar lines, there exists a circle passing through the circum-centres of the four triangles formed by them. In general, given n lines in a plane (no two of which are parallel and no three concurrent) there exists a circle associated with them, such that this circle passes through the centres of the n circles associated with the n sets of $n-1$ lines each, the circle associated with three lines being defined as the circum-circle formed by them.* The special case referred to is when the n lines are tangents to a parabola Γ and then all the circles associated with any sets of lines pass through the focus S of Γ. We modify this result slightly by considering at any stage, not the centres of the system of circles, but points on them which are diametrically opposite to S. Let $a_1, a_2, a_3, \ldots a_n$ denote the n tangents, a_{12} denote the intersection of a_1 and a_2 and a_{123} the point diametrically opposite to S on the circle Q_{123} through a_{12} a_{23}, a_{31}. Then the points $a_{234}, a_{134}, a_{124}, a_{123}$ lie on a circle Q_{1234}, the point on this diametrically opposite to S being denoted by a_{1234} and so on.

2. This may be easily proved, for if the parabola Γ be given by the polar equation

$$l = r(1 + \cos\theta)$$

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* Coolidge: *A Treatise on the Circle and the Sphere* (1916) p. 92, Th. 164.

and the point of contact of the tangent a_i has the angular co-ordinate θ_i, then the equation of the circle Q_{123} is

$$2r\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2} = l\cos\left(\theta - \frac{\theta_1+\theta_2+\theta_3}{2}\right)$$

and the co-ordinates of a_{123} are

$$\left(\frac{l}{2}\sec\frac{\theta_1}{2}\sec\frac{\theta_2}{2}\sec\frac{\theta_3}{2},\ \frac{\theta_1+\theta_2+\theta_3}{2}\right).$$

It may be easily verified that the points a_{234}, a_{134}, a_{124}, a_{123} lie on the circle given by

$$2r\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2}\cos\frac{\theta_4}{2} = l\cos\left(\theta - \frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right),$$

which is the equation to the circle Q_{1234} and so on.

3. Now, let us reciprocate with respect to any circle with S as centre. The three tangents a_1, a_2, a_3 to the parabola Γ and the circle Q_3 go over into three points $\alpha_1, \alpha_2, \alpha_3$ lying on a circle C which passes through the focus S of a parabola g_3 inscribed in $\alpha_1\alpha_2\alpha_3$. The fact that the perpendiculars at a_{23}, a_{31}, a_{12} to Sa_{23}, Sa_{31}, Sa_{12} respectively are concurrent at a_{123} is transformed into the pedal-line theorem for a triangle, the point a_{123} reciprocating into the pedal line of S with respect to $\alpha_1\alpha_2\alpha_3$. Again, since Sa_{123} subtends right angles at a_{234}, a_{134}, a_{124}, the feet of the perpendiculars from S on the pedal lines of S with respect to $\alpha_2\alpha_3\alpha_4, \ldots, \alpha_1\alpha_2\alpha_3$ are in a line which is the reciprocal of a_{234}. Evidently this process of reciprocation can be indefinitely continued and Langley's theorem first stated is therefore true.

4. The following simpler proof* by the use of polar co-ordinates provides us incidentally with a generalization of a known property of the pedal lines of a triangle. For this purpose, it is convenient to adopt a definition. Let the point in which the perpendicular from a point O on a circle C to the pedal line of O with respect to a system of points

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* *Vide* Smith's *Conic Sections*, p. 97 (4).

on C meets the circle again be called the pedal derived or simply the derived point of O with respect to the system of points.

In what follows, the points on the circle C given by the polar equation,

$$r = d \cos \theta$$

(O being taken as the origin) will be invariably named by their angular co-ordinates. Let the n given points be $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n$. The perpendicular from O on the line $\alpha_1\alpha_2$ meets the circle again in the point $\alpha_1 + \alpha_2$. The equation of the line joining this point to α_3 is

$$r \cos (\theta - \alpha_1 - \alpha_2 - \alpha_3) = d \cos \alpha_3 \cos (\alpha_1 + \alpha_2)$$

and that of the line parallel to this through the foot of the perpendicular from O on $\alpha_1\alpha_2$ is

$$r \cos (\theta - \alpha_1 - \alpha_2 - \alpha_3) = d \cos \alpha_1 \cos \alpha_2 \cos \alpha_3.$$

The symmetry of this result in $\alpha_1\alpha_2\alpha_3$ proves the pedal-line-theorem for three points on a circle. Again, the derived point of O with respect to $\alpha_1\alpha_2\alpha_3$ is the point $\alpha_1 + \alpha_2 + \alpha_3$. The line joining this to the point α_4, namely

$$r \cos (\theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) = d \cos \alpha_4 \cos (\alpha_1 + \alpha_2 + \alpha_3)$$

is parallel to the line given by

$$r \cos (\theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) = d \cos \alpha_1 \cos \alpha_2 \cos \alpha_3 \cos \alpha_4$$

which is the pedal line of O with respect to $\alpha_1\alpha_2\alpha_3\alpha_4$. The general theorem is now obvious and we also have the following

THEOREM. The pedal line of O with respect to n points $\alpha_1, \alpha_2, \dots \alpha_n$ on a circle C is parallel to the line joining any one of the points, say α_n, to the derived point of O with respect to the remaining $n - 1$ points $\alpha_1, \alpha_2, \dots \alpha_{n-1}$.

5. The above theorem is a particular case of the following: If the n given points be divided into two groups of p and q points $(p + q = n)$ the pedal line of O with respect to the n points is parallel to the line joining the derived points of O with respect to the two groups separately.

In fact we have the general

THEOREM. If the system of points $a_1\, a_2 \ldots a_n$ be divided into r groups each group containing any number of points whatever, and if $g_1, g_2 \ldots g_r$ be the derived points of O with respect to these groups separately, then the pedal line of O with respect to $a_1, a_2, \ldots a_n$ is parallel to that with respect to $g_1, g_2, \ldots g_r$.

For definiteness, consider the six points $a_1, a_2, \ldots a_6$ divided into three groups

$$a_1\, a_2\, a_3,\ a_4\, a_5,\ a_6.$$

The derived points of O with respect to these groups are given by

$$a_1 + a_2 + a_3,\ a_4 + a_5,\ a_6.$$

The pedal line of O with respect to these three points is

$$r \cos(\theta - a_1 - \ldots - a_6)$$
$$= d \cos(a_1 + a_2 + a_3) \cos(a_4 + a_5) \cos a_6$$

and this is obviously parallel to the pedal line of O with respect $a_1 a_2 \ldots a_6$, namely

$$r \cos(\theta - a_1 - \ldots a_6) = d \cos a_1 \cos a_2 \ldots \cos a_6.$$

The proof is, however, general.

6. Let us now find the equation of the pedal line of a point ϕ with respect to the points $a_1, a_2, \ldots a_n$ on the circle C.

Let the lengths of the perpendiculars from ϕ and the origin O on the pedal line of ϕ with respect to $a_1 a_2 \ldots a_n$ be p' and p and let these perpendiculars make with ϕ S and OS angles a' and a respectively, S being the centre of the circle C. Also, let the line ϕa_r make with ϕS an angle a'_r. Then it is easy to show from a figure that

$$a'_r = a_r + \frac{\pi}{2} - \phi \qquad (r = 1, 2, \ldots n)$$

$$p = p' + d \cos\phi \cos(\pi - \phi - a')$$

and $$a = a' + 2\phi - \pi.$$

Also it is obvious from Art. 4, that

$$\alpha' = \sum_r \alpha'_r$$

and $$p' = d \cos\alpha'_1 \cos\alpha'_2 \ldots \cos\alpha'_n$$

therefore it follows that the equation of the pedal line of ϕ with respect to $\alpha_1\alpha_2 \ldots \alpha_n$ is

$$r \cos(\theta - \alpha) = p$$

where $$\alpha = \sum_r \alpha_r - (n-2)\phi + (n-2)\pi/2$$

and $$p = d \prod_r \sin(\phi - \alpha_r) \ldots \ldots$$

$$+ d \cos\phi \cos[\Sigma \alpha_r - (n-1)\phi + (n-2)\pi/2].$$

7. As an immediate inference from the above equation we have the following

THEOREM. If the point ϕ moving on the circle describes an angle ω about the centre, the corresponding pedal line turns through an angle $-(n-2)\omega/2$.

The case $n = 3$ is well-known. We may state further that the pedal lines of diametrically opposite points are either parallel or perpendicular according as n is even or odd.

There are $n - 2$ pedal lines parallel to a given line.

The derived point of ϕ with respect to $\alpha_1\alpha_2 \ldots \alpha_n$ is the point given by

$$\sum_r \alpha_r - (n-1)(\phi - \pi/2).$$

8. The concept of the pedal angle may also be generalised. If there be two systems A and B of points $\alpha_1, \alpha_2, \ldots \alpha_n$ and $\beta_1, \beta_2, \ldots \beta_n$, respectively, on the circle, the angle between the pedal lines of ϕ with respect to A and B is, from Art. 6

$$\psi = \sum_r \alpha_r - \sum_r \beta_r$$

The angle ψ which is independent of ϕ may be defined as the 'Pedal angle'* between the two systems A and B. In particular, if $\psi = 0$ or $\pi/2$ (mod π) the two systems may be said to be pedo-parallel or pedo-perpendicular, respectively. It is necessary that the two systems should contain the same number of points.

The theorem of Art. 5 may now be stated thus: If the system of points $\alpha_1, \alpha_2, \dots \alpha_n$ be divided into r groups in two different ways and the derived points of an arbitrary fixed point on the circle be taken for each group, the two sets, each of r points so obtained are pedo-parallel.

A variety of other theorems may be stated. For instance, let there be three systems A, B, C each of n points $\alpha_1, \alpha_2, \dots \alpha_n$, $\beta_1 \beta_2 \dots \beta_n$, $\gamma_1, \gamma_2, \dots \gamma_n$ respectively, such that γ_r is the derived point of α_r with respect to B. If the pedal-angle between A and B be θ, prove that the pedal angle between A and C is $n\theta - \frac{n(n-1)}{2}\pi$.

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A. RANGANATHA RAO.

A new method for obtaining the product formula for sin x.

The following method appears to be very simple, and it is based nly on the supposition that a function $f(x)$ increases or decreases according as

$$f'(x) \gtrless 0$$

and no double limit operation as is usually employed need be used. Moreover something more than the product formula itself is proved in this note.

The proof consists in showing that

$$\text{(i)} \quad f(x) = \frac{\sin \pi x}{x\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right) \dots \left(1 - \frac{x}{n}\right)} \quad \text{where } n = 2^r$$

* The concept of the 'pedal angle' is due to Dr. R. Vaidyanathaswamy who uses it in connection with two inscribed triangles of a circle.

increases in the range $(0, \frac{1}{2})$ and decreases in the range $(\frac{1}{2}, 1)$ and that

(ii) $$\phi(x) = \frac{\sin \pi x}{x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\dots\left(1-\frac{x^2}{n^2}\right)}$$

where $n = 2^r$, is always a decreasing function for all positive values of x. By means of these results, we conclude that $\sin \pi x$ lies between

$$\pi x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\dots\dots\left(1-\frac{x^2}{n^2}\right),$$

and $$\pi x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\dots\dots\left(1-\frac{x}{n}\right),$$

where $n = 2^r$, and $0 \leqslant x \leqslant 1$. The extension for other values of x follows from the periodicity of $\sin x$ and the Infinite product

Proof of (i) :—

Now $$\sin \pi x = 2 \sin \frac{\pi x}{2} \cos \frac{\pi x}{2}$$

$$= 2 \sin \frac{\pi}{2} x \sin \frac{\pi(1-x)}{2}.$$

Operating this r times, we obtain

$$\sin \pi x = 2^k \sin \frac{\pi x}{2n} \sin \pi\left(\frac{1}{2n}-\frac{x}{2n}\right) \sin \pi\left(\frac{1}{2n}+\frac{x}{2n}\right) \times$$

$$\sin \pi\left(\frac{2}{2n}-\frac{x}{2n}\right) \sin \pi\left(\frac{2}{2n}+\frac{x}{2n}\right) \times$$

$$\sin \pi\left(\frac{3}{2n}-\frac{x}{2n}\right) \sin \pi\left(\frac{3}{2n}+\frac{x}{2n}\right) \times$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$\sin \pi\left(\frac{n}{2n}-\frac{x}{2n}\right) \sin \pi\left(\frac{n}{2n}+\frac{x}{2n}\right)$$

where $n = 2^r$, and k a positive integer.

$$\therefore \quad f(x) = 2^k . \frac{\sin \frac{\pi x}{2n}}{x} . \frac{\sin \pi\left(\frac{1}{2n}-\frac{x}{2n}\right)}{1-x} . \frac{\sin \pi\left(\frac{1}{2n}+\frac{x}{2n}\right)}{1+x} .$$

$$\frac{\sin \pi\left(\frac{2}{2n}-\frac{x}{2n}\right)}{1-\frac{x}{2}} \dots\dots \frac{\sin \pi\left(\frac{1}{2}-\frac{x}{2n}\right)}{1-\frac{x}{n}}.$$

Now if we put $\psi(t) = \log \dfrac{\sin \pi t}{t}$,

$$\begin{aligned}\log f(x) = \text{a constant} &+ \left[\psi\left(\frac{x}{2n}\right) + \psi\left(\frac{1}{2n} - \frac{x}{2n}\right)\right] \\ &+ \left[\psi\left(\frac{2}{2n} + \frac{x}{2n}\right) + \psi\left(\frac{3}{2n} - \frac{x}{2n}\right)\right] \\ &+ \quad \cdots \qquad \cdots \qquad \cdots \\ &+ \left[\psi\left(\frac{3}{2n} + \frac{x}{2n}\right) + \psi\left(\frac{4}{2n} - \frac{x}{2n}\right)\right] \\ &+ \left[\psi\left(\frac{n-1}{2n} + \frac{x}{2n}\right) + \psi\left(\frac{n}{2n} - \frac{x}{2n}\right)\right].\end{aligned}$$

We will show that for $0 \leqslant x \leqslant \frac{1}{2}$, each one of the functions in the brackets is an increasing function.

Now

$$\psi(t) = \log \frac{\sin \pi t}{t};$$

$$\psi'(t) = \pi \cot \pi t - \frac{1}{t} < 0;$$

$$\psi''(t) = -\pi^2 \operatorname{cosec}^2 \pi t + \frac{1}{t^2};$$

$$= \frac{1}{t^2} - \frac{\pi^2}{\sin^2 \pi t}$$

which is always negative.

Hence $\psi'(t)$ is negative and decreases monotonically as t increases.

Now consider

$$\lambda(x) = \psi\left(\frac{r}{2n} + \frac{x}{2n}\right) + \psi\left(\frac{r+1}{2n} - \frac{x}{2n}\right).$$

$$\therefore \quad \lambda'(x) = \frac{1}{2n}\left\{\psi'\left(\frac{r}{2n} + \frac{x}{2n}\right) - \psi'\left(\frac{r+1}{2n} - \frac{x}{2n}\right)\right\}$$

and if $0 \leqslant x \leqslant \frac{1}{2}$,

$$\frac{r+1}{2n} - \frac{x}{2n} > \frac{r}{2n} + \frac{x}{2n}.$$

Hence $\lambda'(x)$ is positive for $0 < x < \frac{1}{2}$ and negative for $\frac{1}{2} \leqslant x \leqslant 1$, as $\frac{r+1}{2n} - \frac{x}{2n} < \frac{r}{2n} + \frac{x}{2n}$ for those values of x. Hence $\log f(x)$ satisfies the required property.

$$\therefore \qquad \frac{\sin \pi x}{x(1-x^2/1^2)(1-x^2/2^2)\ \ldots\ldots\ (1-x/n)}$$

$(n = 2^r)$ is an increasing function of x in the range $(0, \frac{1}{2})$, and decreasin range $(\frac{1}{2}, 1)$.

Or it shows that

$$\sin \pi x > \pi x \left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\ldots\ldots\left(1-\frac{x}{n}\right) \ldots \quad (3)$$

where $n = 2^r$, as the minimum value of

$$\frac{\sin \pi x}{x(1-x^2/1^2)(1-x^2/2^2)\ \ldots\ldots\ (1-x/n)}$$

in the range $(0, 1)$ is attained at either $x = 0$ or 1,

Proof of (2) :—

$$\phi(x) = \frac{\sin \pi x}{x(1-x^2/1^2)(1-x^2/2^2)\ \ldots\ldots\ (1-x^2/n^2)} \text{ where } n = 2_r.$$

With the same notation we obtain

$$\log \phi(x) = \text{a constant} + \psi\left(\frac{x}{2n}\right)$$

$$+\left[\psi\left(\frac{1}{2n}-\frac{x}{2n}\right)+\psi\left(\frac{1}{2n}+\frac{x}{2n}\right)\right]$$

$$+\left[\psi\left(\frac{2}{2n}-\frac{x}{2n}\right)+\psi\left(\frac{2}{2n}+\frac{x}{2n}\right)\right]+\ldots\ldots$$

$$+\left[\psi\left(\frac{1}{4}-\frac{x}{4n}\right)+\psi\left(\frac{1}{4}+\frac{x}{4n}\right)\right]$$

which is obtained by expressing the last term, *viz.*, $\sin \pi\left(\frac{1}{2}-\frac{x}{2n}\right)$ in the form $2 \sin \pi\left(\frac{1}{4}-\frac{x}{4n}\right) \sin \pi\left(\frac{1}{4}+\frac{x}{2n}\right)$.

We have already proved that $\psi'(x)$ is negative and monotonically decreasing. Hence $\psi(x/2n)$ always decreases, and if we write

$$\mu(x) = \psi\left(k-\frac{x}{2n}\right)+\psi\left(k+\frac{x}{2n}\right),$$

then

$$\mu'(x) = \frac{1}{2n}\left[-\psi'\left(k-\frac{x}{2n}\right) + \psi'\left(k+\frac{x}{2n}\right)\right]$$

and as $k + \frac{x}{2n} > k - \frac{x}{2n}$ for positive values of x, $\mu'(x)$ is negative and so each one of these functions is also decreasing. Hence $\phi(x)$ is a decreasing function for positive values of x, and as $\phi(-x) = \phi(x)$, $\phi(x)$ also decreases as x decreases to $-\infty$.

This shows that

$$\phi(x) = \frac{\sin \pi x}{x\left[1-\frac{x^2}{1^2}\right]\left[1-\frac{x^2}{2^2}\right]\cdots\left[1-\frac{x^2}{n^2}\right]}$$

where $n = 2^r$, is a decreasing function in $0 \leqslant x \leqslant 1$.

Hence

$$\sin \pi x < \pi x \left[1-\frac{x^2}{1^2}\right]\left[1-\frac{x^2}{2^2}\right]\cdots\left[1-\frac{x^2}{n^2}\right] \quad \cdots \quad (4)$$

as the maximum value is obviously at $x = 0$.

From (3) and (4) it follows that $\sin \pi x$ lies between

$$\pi x\left[1-\frac{x^2}{1^2}\right]\left[1-\frac{x^2}{2^2}\right]\left[1-\frac{x^2}{3^2}\right]\cdots\left[1-\frac{x^2}{n^2}\right]$$

and $$\pi x\left[1-\frac{x^2}{1^2}\right]\left[1-\frac{x^2}{2^2}\right]\cdots\left[1-\frac{x}{n}\right].$$

and as $1 + \frac{x}{n} \to 1$ as n tends to infinity, we obtain the Infinite Product for the range $0 \leqslant x \leqslant 1$; and for other value of x, the product formula follows from the periodicity of $\sin x$ and the Infinite Product.

K. VENKATACHALA AIYANGAR.

Lecture Notes.

(1) Geodesics.

The object of this note is to point out a useful variant of the differential equation of a geodesic on a surface, and to work out is implications.

1. The parametric curves being orthogonal and non-geodesic, so that, in the notation of Forsyth,* $F=0$, a geodesic is given by

$$2\frac{d}{ds}\left\{E\frac{du}{ds}\right\}=E_1\left(\frac{du}{ds}\right)^2+G_1\left(\frac{dv}{ds}\right)^2 \qquad \dots \quad (1)$$

Since $v\dfrac{du}{ds}\neq 0$, by hypothesis, we multiply both sides of (1) by $v\dfrac{du}{ds}$ and add $\left(\dfrac{du}{ds}\right)^2\left(E\dfrac{du}{ds}-v\dfrac{dE}{ds}\right)$

to each side; then the resulting equation reduces to

$$\frac{d}{ds}\left\{Ev\left(\frac{du}{ds}\right)^2\right\}=\frac{du}{ds}\cdot\frac{dv}{ds}\left\{\frac{du}{ds}(E-vE_2)+G_1v\frac{dv}{ds}\right\} \qquad \dots \quad (2)$$

which is a variant of (1) but is not equivalent to (1) as it includes the factor $v\dfrac{du}{ds}$.

If θ be the angle which a geodesic makes with the parametric curve $v=$a constant,

$$\cos\theta=E^{\frac{1}{2}}\frac{du}{ds} \text{ and } \sin\theta=G^{\frac{1}{2}}\frac{dv}{ds};$$

the left-hand side of (2) may be replaced by $\dfrac{d}{ds}(v\cos^2\theta)$.

Similarly, from the cognate equation

$$2\frac{d}{ds}\left\{G\frac{dv}{ds}\right\}=E_2\left(\frac{du}{ds}\right)^2+G_2\left(\frac{dv}{ds}\right)^2$$

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* *Lectures on the Differential Geometry of Curves and Surfaces* (1912), Chapter II,

we derive the variant

$$\frac{d}{ds}\left\{ Gu\left(\frac{dv}{ds}\right)^2 \right\} = \frac{du}{ds}\frac{dv}{ds}\left\{ \frac{dv}{ds}(G - u\,G_1) + E_2 u \frac{du}{ds} \right\} \quad \ldots \quad (3)$$

which differs from (1) in the inclusion of the irrelevant factor $u\dfrac{dv}{ds}$.

Adding the equations (2) and (3), we get a third variant

$$\frac{d}{ds}\left\{ Ev\left(\frac{du}{ds}\right)^2 + Gu\left(\frac{dv}{ds}\right)^2 \right\}$$

$$= \frac{du}{ds}\frac{dv}{ds}\left[\frac{du}{ds}(E + \overline{u - v}\,.\,E_2) + \frac{dv}{ds}(G + \overline{v - u}\,G_1) \right] \quad \ldots \quad (4)$$

where the left-hand side may be replaced by

$$\frac{d}{ds}(u \sin^2 \theta + v \cos^2 \theta)$$

From the mode of derivation of (4), it is evident that it is equivalent to

$$v\frac{du}{ds}\left[2\frac{d}{ds}\left\{ E\frac{du}{ds} \right\} - E_1\left(\frac{du}{ds}\right)^2 - G_1\left(\frac{dv}{ds}\right)^2 \right]$$

$$+ u\frac{dv}{ds}\left[2\frac{d}{ds}\left\{ G\frac{dv}{ds} \right\} - E_2\left(\frac{du}{ds}\right)^2 - G_2\left(\frac{dv}{ds}\right)^2 \right] = 0 \quad \ldots \quad (5)$$

Now differentiating the identity

$$E\left(\frac{du}{ds}\right)^2 + G\left(\frac{dv}{ds}\right)^2 = 1$$

with respect to s, we get

$$\left(\frac{du}{ds}\right)^2\left(E_1\frac{du}{ds} + E_2\frac{dv}{ds}\right) + \left(\frac{dv}{ds}\right)^2\left(G_1\frac{du}{ds} + G_2\frac{dv}{ds}\right)$$

$$+ 2E\frac{du}{ds}\cdot\frac{d^2u}{ds^2} + 2G\frac{dv}{ds}\frac{d^2v}{ds^2} = 0$$

which may be put in the form

$$\frac{du}{ds}\left[2\frac{d}{ds}\left\{ E\frac{du}{ds} \right\} - E_1\left(\frac{du}{ds}\right)^2 - G_1\left(\frac{dv}{ds}\right)^2 \right]$$

$$+ \frac{dv}{ds}\left[2\frac{d}{ds}\left\{ G\frac{dv}{ds} \right\} - E_2\left(\frac{du}{ds}\right)^2 - G_2\left(\frac{dv}{ds}\right)^2 \right] = 0 \quad \ldots \quad (6)$$

From (5) and (6) it is clear that (4) includes, besides the geodesics, the curve $v - u = 0$ and the parametric curves.

Cor.: (1) Carrying out the differentiation on the first member of (2), and cancelling the common terms on both sides, we get

$$\frac{d}{ds}\left\{ E\left(\frac{du}{ds}\right)^2 \right\} = \frac{du}{ds} \cdot \frac{dv}{ds} \left(G_1 \frac{dv}{ds} - E_2 \frac{du}{ds} \right) \quad \dots \quad (7)$$

which includes only one spurious factor $\frac{du}{ds}$.

Hence, since $\frac{du}{ds} = \cos\theta \,.\, E^{-\frac{1}{2}}$ and $\frac{dv}{ds} = G^{-\frac{1}{2}} \sin\theta$, (**7**) reduces to

$$-2\cos\theta\sin\theta\frac{d\theta}{ds} = \cos\theta\sin\theta\,(EG)^{-\frac{1}{2}}\left(G_1\frac{dv}{ds} - E_2\frac{du}{ds}\right)$$

i.e.

$$E^{\frac{1}{2}} G^{\frac{1}{2}} \frac{d\theta}{ds} = \tfrac{1}{2}\left(E_2 \frac{du}{ds} - G_1 \frac{dv}{ds}\right)^{*} \quad \dots \quad (8)$$

Cor. ( ) If $E + (u - v) E_2 = 0$ and $G + (v - u) G_1 = 0$, as in the case of planes, spheres, and central quadrics, we get, at once from (4).

$$\frac{d}{ds}\left(u \sin^2\theta + v\cos^2\theta\right) = 0$$

i.e., $$u\sin^2\theta + v\cos^2\theta = \text{a constant},\dagger \quad \dots \quad (9)$$

along a geodesic, the parametric curves being, of course, properly chosen.

2. Conversely, if $u\sin^2\theta + v\cos^2\theta = a$ be a geodesic on a surface referred to orthogonal parametric curves $u =$ constant and $v =$ constant, we should have simultaneously

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* This result may be identified with Forsyth's result on p. 148, (*Differential Geometry*).

† This is Liouville's form of the well known Theorem "pD is constant for geodesics on quadrics," due to Joachimsthal, (*Vide* p. 427, A Treatise on Analytic Geometry of Three Dimensions by G. Salmon, revised by R. A. P. Rogers, Longmans, 1914).

$$\left.\begin{aligned} &E\left(\frac{du}{ds}\right)^2 + G\left(\frac{dv}{ds}\right)^2 = 1 \\ &E v\left(\frac{du^2}{ds}\right) + G u\left(\frac{dv}{ds}\right)^2 = a \\ &\{E + (u-v)E_2\}\frac{du}{ds} + \{G+(v-u)G_1\}\frac{dv}{ds} = 0. \end{aligned}\right\} \quad \dots \quad (10)$$

Eliminating $\frac{du}{ds}$ and $\frac{dv}{ds}$ from (10), we have

$$G(u-a)(E + \overline{u-v}\,E_2)^2 + E(v-a)(G + \overline{v-u}\,G_1)^2 = 0. \quad \dots (11)$$

If (11) is not identically satisfied, the geodesic

$$u \sin^2\theta + v\cos^2\theta = a$$

does not generally exist.

Again, if $u \sin^2\theta + v\cos^2\theta =$ a constant for every geodesic on a surface, then $E + (u-v)E_2$ and $G+(v-u)G_1$ must vanish separately. Otherwise, only one geodesic can be drawn through a point (u, v), since $\frac{dv}{du}$ has a definite value given by

$$-\frac{E + (u-v)E_2}{G + (v-u)G_1}$$

at the point. This is certainly absurd, for a geodesic must lie in every direction on the surface.

Thus, Liouville's surfaces given by

$$ds^2 = (v-u)\,e(u)\,du^2 + (u-v)\,g(v)\,dv^2$$

are the only surfaces for which the property (9) holds for all geodesics; $e(u)$ and $g(v)$ being functions of u and v obtained from the equations

$$E + (u-v)E_2 = 0 \text{ and } G + (v-u)G_1 = 0$$

i.e.,

$$E = (v-u)\,e(u) \text{ and } G = (u-v)\,g(v).$$

A. A. KRISHNASWAMI AYYANGAR.

(2) On the foci and axes of a conic.

In the last two issues of the *J.I.M.S.*, have appeared some notes on the focus of a central conic, based on the relation of the circular points at infinity to the foci. In this connection, I wish to point out a formula* given by Casey, which seems to be best fitted for numerical work. By means of this formula, we can simultaneously obtain the co-ordinates of the foci and the lengths of the axes.

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be a central conic. Transfer the origin to the centre, the direction of the axes remaining the same; and the equation reduces to

$$ax^2 + 2hxy + by^2 + c' = 0 \qquad \dots \quad (1)$$

where $c' = \Delta/C$ with the usual notation.

Let (X, Y) be a focus of this conic. By central symmetry (— X, — Y) is the complementary focus and the product of the perpendiculars from these foci on a tangent is equal to a constant, say λ, the square of a semi-axis of the conic.

Let $\xi = ax_1 + hy_1, \quad \eta = hx_1 + by_1,$

so that

$$x_1 = \frac{b\xi - h\eta}{ab - h^2}, \quad y_1 = \frac{h\xi - a\eta}{h^2 - ab}$$

If (x_1, y_1) be a point on (1), ξ, η satisfy the equation

$$a(b\xi - h\eta)^2 - 2h(b\xi - h\eta)(h\xi - a\eta) + b(h\xi - a\eta)^2 + c'(ab - h^2)^2 = 0$$

i.e.,

$$b\xi^2 - 2h\xi\eta + a\eta^2 = c'(h^2 - ab). \qquad \dots \quad (2)$$

The product of the perpendiculars from (X, Y), (— X, — Y) on the tangent at (x_1, y_1) to (1) is given by

$$\frac{c'^2 - (X\xi + Y\eta)^2}{\xi^2 + \eta^2} = \lambda$$

so that

$$\xi^2(X^2 + \lambda) + 2\xi\eta XY + \eta^2(Y^2 + \lambda) = c'^2 \qquad \dots \quad (3)$$

Comparing (2) and (3), we have

$$\frac{X^2 + \lambda}{b} = \frac{Y^2 + \lambda}{a} = \frac{XY}{-h} = \frac{c'}{h^2 - ab} \qquad \dots \quad (A)$$

and hence, each is equal to $\dfrac{X^2 - Y^2}{b - a}$

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* *Vide* p. 533, Exercise No. 74, *A Treatise on Analytical Geometry*, by John Casey, 2nd Edition, 1893.

Thus, the foci are given by

$$X^2 - Y^2 = \frac{(b-a)c'}{h^2 - ab}, \quad XY = \frac{c'h}{ab - h^2} \qquad \ldots \quad (4)$$

which can be readily solved in any numerical example.

The value of λ is obtained from

$$\lambda = \frac{bc'}{h^2 - ab} - X^2 = -\frac{b\Delta}{C^2} - X^2 \qquad \ldots \quad (5)$$

Thus, as soon as X is known, λ is also known.

Again, eliminating X, Y from the equations (A), we get the well-known equation giving the squares of the semi-axes,

$$\left[a + \frac{c'}{\lambda}\right]\left[b + \frac{c'}{\lambda}\right] = h^2$$

i.e.,

$$\left[a + \frac{\Delta}{C\lambda}\right]\left[b + \frac{\Delta}{C\lambda}\right] = h^2. \qquad \ldots \quad (6)$$

Numerical Illustration.

To find the foci and the lengths of the axes of the conic

$$y^2 - 4xy - 5x^2 + 6y + 42x - 63 = 0.$$

The centre of this conic is (3, 3) and the equation reduced to the centre is

$$y^2 - 4xy - 5x^2 + 9 = 0.$$

The foci are given by $X^2 - Y^2 = 6$, $XY = 2$,

so that

$$X^2 = 3 \pm \sqrt{13}, \qquad X = \pm\sqrt{3 + \sqrt{13}}, \ \pm\sqrt{3 - \sqrt{13}}.$$

The co-ordinates of the real foci are

$$(3 \pm \sqrt{3 + \sqrt{13}}, \ 3 \pm \sqrt{-3 + \sqrt{13}}),$$

and those of the imaginary foci are

$$(3 \pm \sqrt{3 - \sqrt{13}}, \ 3 \pm \sqrt{-3 - \sqrt{13}})$$

The squares of the semi-axes are $-2 \pm \sqrt{13}$.

A. A. KRISHNASWAMI AYYANGAR.

Solutions.

Question 1521.

(A. NARASINGA RAO) :—(1) If from a point on the circum-circle of a triangle lines be drawn making the same angle θ with the sides taken one way round, the corresponding points of intersection lie on a line—the Poncelet line of the point. Show that the envelope of the Poncelet lines of a triangle is a tricusp hypocycloid touching the sides.

(2) Given any tricusp hypocycloid, show that every triangle circumscribed to it is a Poncelet triangle (*i.e.*) one whose Poncelet lines touch the tricusp and that the corresponding pedal angle θ may be found from any one of the following properties:—

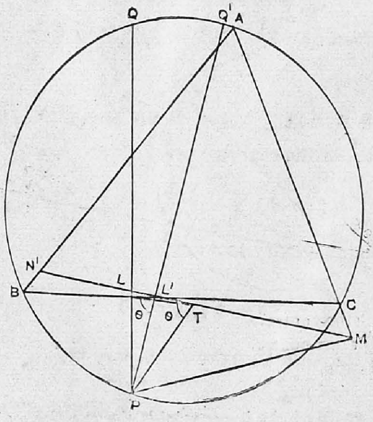
(1) The sides of the triangle make with any of the cusp-tangents angles whose sum is θ (mod. π).

(2) The circum-centre and ortho-centre of the triangle subtend at the centre of the tricusp an angle 2θ.

(3) The third tangent from each vertex of the triangle makes an angle θ with the opposite side.

Solution by B. Ramamurti.

(1) The first part of this,—that the points of intersection lie on a line—is quite easy to prove.



Let PL′, PM′, PN′ be drawn making the same angle θ with the sides BC, CA, AB respectively and PL, PM, PN perpendicular to the sides.

Let PL, PL′ meet the circle again in Q and Q′.

It can be easily proved that LMN, and L′M′N′ are collinear and that LMN, L′M′N′ are respectively parallel to AQ and AQ′ so that

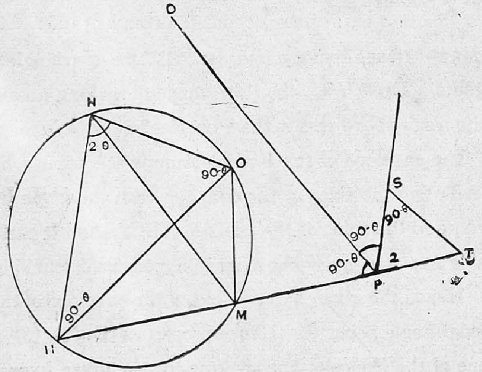
$$\angle LTL' = \angle QAQ' = \angle QPQ' = \angle LPL' = 90^\circ - \theta.$$

$\therefore$ LL′TP are cyclic.

Hence $\angle PTL' = 90^\circ$ and $\angle PTL = \theta$.

Now consider the parabola touching the sides and having P as the focus. Then LMN, the pedal line of P, is the tangent at the vertex; and since the perpendicular from the focus P on L′M′N′ lies on this, L′M′N′ is a tangent to this parabola making a constant angle $= 90^\circ - \theta$ with the tangent at the vertex. Hence by the result of Q. 417 the envelope of the Poncelet line is a tricusp hypocycloid touching the sides.

[Q. 417. If parabolas are escribed to a triangle, show that any system of corresponding lines connected with the figure envelopes a tricusp hypocycloid].



H—Ortho-centre.
O—Circum-centre.
S—Focus.
HT—Directrix.
N—Centre of the hypocycloid, the envelope of PQ.

Hence in the figure of Q. 417 $\angle QPM = 90^\circ - \theta = \angle QPS$.

$$\therefore \qquad \angle PST = \angle PTS = 90^\circ - \theta.$$

$$\therefore \qquad \angle SPT = \angle HNO = 180^\circ - 2\,(90^\circ - \theta) = 2\theta.$$

or the circum- and crtho-centres of the triangle subtend at the centre of the tricusp an angle $= 2\theta$. ... (2)

Again the third tangent from A should be the Poncelet line of A which is AA_1 making an angle θ with BC;

or the third tangent from each vertex of the triangle makes an angle θ with the opposite side. ... (1)

Again according to the result of Q. 1376,

$$\theta = \theta_1 + \theta_2 + \theta_3 - 2n\pi$$

or
$$\theta_1 + \theta_2 + \theta_3 = 2\pi + \theta,$$

i.e., the sides of the triangle make with any of the cusp tangents angles whose sum is θ (mod. 2π). ... (3)

Given any tricusp hypocycloid, let ABC be a triangle formed by three tangents. Draw AA', the third tangent, making an angle θ with the opposite side. Now this is the Poncelet line of A, the pedal angle being θ. The envelope of the Poncelet line is the tricusp hypocycloid touching AA_1 and the sides of the triangle. But the given hypocycloid touches AA_1 and the sides of the triangle ABC. And it can be proved that there is one and only one tricusp hypocycloid touching AA_1 and the sides. Hence the given hypocycloid is the envelope of the Poncelet line, the pedal angle being θ. Hence properties (1), (2), (3) proved for the envelope of the Poncelet line are true for the given hypocycloid.

Question 1548.

(S. Narayana Aiyar, M.A.) :—Establish the following relations :—

(1) $$\frac{\operatorname{sn} 3u}{\operatorname{sn} u \operatorname{cn} 2u + \operatorname{sn} 2u \operatorname{cn} u} = \frac{1 + \operatorname{dn} 3u}{\operatorname{dn} u + \operatorname{dn} 2u}$$

(2) $$\frac{\operatorname{sn} 3u}{\operatorname{sn} u \operatorname{dn} 2u + \operatorname{sn} 2u \operatorname{dn} u} = \frac{1 + \operatorname{cn} 3u}{\operatorname{cn} u + \operatorname{cn} 2u}.$$

*Solution by K. V. Krishna Sastry, M.A.**

Edwards, in his *Integral Calculus*, Vol. II, has givent he following example on p. 523, (Ex. 27 (i)),

Prove that

$$\frac{1 + \operatorname{dn}(u+v)}{\operatorname{sn}(u+v)} = k^2 \frac{\operatorname{sn} u \operatorname{cn} v - \operatorname{sn} v \operatorname{cn} u}{\operatorname{dn} v - \operatorname{dn} u}.$$

This can be verified easily.

Now remembering that

$$\operatorname{dn}^2 v - \operatorname{dn}^2 u = k^2 (\operatorname{sn}^2 u \operatorname{cn}^2 v - \operatorname{sn}^2 v \operatorname{cn}^2 u)$$

the above relation may be written

$$\frac{1 + \operatorname{dn}(u+v)}{\operatorname{sn}(u+v)} = \frac{\operatorname{dn} v + \operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} v + \operatorname{sn} v \operatorname{cn} u};$$

i.e., $$\frac{\operatorname{sn}(u+v)}{\operatorname{sn} u \operatorname{cn} v + \operatorname{sn} v \operatorname{cn} u} = \frac{1 + \operatorname{dn}(u+v)}{\operatorname{dn} u + \operatorname{dn} v}. \qquad \ldots \text{(i)}$$

Similarly, it can be shown that

$$\frac{\operatorname{sn}(u+v)}{\operatorname{sn} u \operatorname{dn} v + \operatorname{sn} v \operatorname{dn} u} = \frac{1 + \operatorname{cn}(u+v)}{\operatorname{cn}^{n} + \operatorname{cn} v}. \qquad \ldots \text{(ii)}$$

The required results are only particular cases respectively of (i) and (ii), if we write $v = 2u$.

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* For another Solution, *vide*, *J.I.M* S., August 1932, *p. 244*,

Question 1550.

(PROF. K. J. SANJANA) :—Let n be an odd prime and N denote the number $2^n - 1$.

(1) When N is also a prime, prove that $N - 1$ is a multiple of $2n$.

(This readily follows from Fermat's theorem).

(2) When N is the product of two factors A and B, prove that $A \backsim B$ is a multiple of $2n$.

Solution by P. Jagannadan.

Euler has proved that when P is an odd prime and x any positive integer, $\dfrac{x^P - 1}{x - 1}$ has got besides P prime factors only of the form $(m\,P + 1)$ where m is a positive integer.

Here $2^n - 1$ will not be divisible by n. Since $(2, n) = 1$, we have

$$2^{\phi(n)} \equiv 1 \text{ (mod. } n), \text{ i.e., } 2^{n-1} \equiv 1 \text{ (mod. } n) \text{ or } 2^n \equiv 2 \text{ (mod. } n)$$

so that $2^n \not\equiv 1$ (mod. n).

Now $2^n - 1$ will have prime factors only of the form $(mn + 1)$, so that every factor of $2^n - 1$ will be only of the form $(mn + 1)$.

Let $A = (m_1 n + 1)$ and $B = (m_2 n + 1)$ where $N = AB$.

Since A and B are odd, m_1 and m_2 should be even.

Now $A \backsim B$, *i.e.*, $(m_1 n + 1) \backsim (m_2 n + 1) = n\,(m_1 \backsim m_2)$ which is evidently a multiple of $2n$.

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Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their Questions.

1633. (R. VAIDYANATHASWAMY) :—Prove that

$$\tan\frac{a+b+c-d}{2}+\frac{\cos a+\cos b+\cos c}{\sin a+\sin b+\sin c}=0$$

is a symmetric relation between a, b, c, d. Prove further that the two relations,

$$\tan\frac{a+b+c+d+\pi-2e}{2}+\frac{\cos a+\cos b+\cos c+\cos d}{\sin a+\sin b+\sin c+\sin d}=0,$$

and the similar equation, in which d and e are interchanged, constitute two symmetric conditions on a, b, c, d, e.

Indicate the generalisation to n numbers $a, b, c, \ldots \ldots$

1634. (J. A. DEVASAGAYAM, M.A.) :—The three conics having a common focus and inscribed in the three parallelograms formed by any three pairs of parallel straight lines have concurrent common chords.

1635. (K. J. SANJANA, M.A.) :—(i) n distinct primes of the form $4m+1$ are multiplied together; show that the product can be resolved into a sum of two squares in 2^{n-1} ways.

(ii) Prove that the nth power of a prime of the form $4m+1$ can be expressed as a sum of two squares in $\frac{1}{2}(n+1)$ or $\frac{1}{2}n$ ways according as n is an odd or an even integer.

1636. (M. N. RAMAKRISHNA PILLAI) :—Trace the curve

$$y=(k/\log x)^x$$

for different values of k.

1639. (A. A. KRISHNASWAMY AYYANGAR) :—Two conics have double contact at U, W and V is the common pole of UW. ABC is a triangle inscribed in one conic and PQR a triangle inscribed in the other such that P lies on BC and AB, AC meet PQ, PR respectively on UW. Prove that the six points U, V, W, A, Q, R lie on a conic.

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during the months of October and November 1932.

1 Acta Mathematica, **59**, 3, 4.
2 American Journal of Mathematics, **54**, 4 (3 copies).
3 American Mathematical Monthly, **39**, 7, 8 (3 copies each).
4 Annales de l'Ecole Normale Superieure, Aug. Sep. Oct. 1932, (2 copies each).
5 Annals of Mathematics, **33**, 4 (2 copies).
6 Astrophysical Journal, **76**, 3.
7 Bulletin des Sciences Mathematiques, **56**, July, Aug., Sep., and Oct. 1932 (2 copies each).
8 Bulletin of the American Mathematical Society, **38**, 1, 2.
9 Crelle's Journal, **168**, 3.
10 Current Science, **1**, 4, 5.
11 Jahresbericht der dentschen mathematiker Vereinigung. **42**, 1—4.
12 Japanese Journal of Mathematics, **9**, 2.
13 Mathematical Gazette, **16**, 220 (2 copies).
14 Mathematische Annalen, **107**, 3.
15 Monthly Notices of the Royal Astronomical Society, **92**, 9.
16 Philosophical Magazine, **14**, 91, 92.
17 Philosophical Transactions of the Royal Society, **231**, 75, 145.
18 Popular Astronomy, **40**, 8.
19 Proceedings of the Edinburgh Mathematical Society, **3**, 2 (3 copies).
20 Proceedings of the London Mathematical Society, **34**, 4 (3 copies).
21 Proceedings of the Physico Mathematical Society of Japan, **14**, 8.
22 Proceedings of the Royal Society of London **138**, 834, 835.
23 Quarterly Journal of Mathematics, **3**, 11.
24 Revista Mathematica Hispano Americana **7**, 3, 4, 5, 6.
25 Tohoku Mathematical Journal, **36**, 1.

Booklet.

Mathematical Notes, No. 27, by Aitken, M.A., D.Sc., F.R.S.E.

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